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1993 J. Phys. A: Math. Gen. 26 6027

(http://iopscience.iop.org/0305-4470/26/21/039)

J. Phys. A: Math. Gen. 26 (1993) 6027–6031. Printed in the UK

Travelling wave solutions to the two-dimensional KdV–Burgers equation

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Received 26 May 1993, in final form 14 July 1993

Abstract. Travelling wave solutions for the two-dimensional KdV–Burgers equation are obtained by using *Mathematica*. They are shown to be sums of a shock wave and solitary wave.

The KdV–Burgers equation is a nonlinear partial differential equation which arise in the study of many physical problems [1, 2]. Recently it has received much attention [3–5]. In this paper, we use a computer algebra system to search for appropriate exact solutions to the 2D KdV–Burgers equation. Travelling wave solutions are obtained. We prove that the solutions can be expressed as a sum of shock wave solutions of a 2D Burgers equation and solitary wave solutions of a 2D KdV equation. At the time of writing the authors have not seen two-dimensional results like these in the literature. Although Barrera and Brugarino [6] discussed the similarity solutions of the 2D KdV–Burgers equation using Lie group analysis and examined some features of these invariant solutions, they could not obtain the *exact* travelling wave solution.

We attempt to find the exact solution to the 2D KdV–Burgers equation of the following form:

$$(u_t + uu_x - mu_{xx} + nu_{xxx})_x + su_{yy} = 0 \tag{1}$$

where $u = u(x, y, t)$ is a function of x, y and t, coefficients m, n and s are real constants. Equation (1) is an extension of the KdV–Burgers equation for the two-dimensional case just as in the relationship between the KdV equation and the KP equation.

Murakami [7] showed that the Hopf–Cole transformation

$$u = [\log(f)]_x \tag{2}$$

can reduce the 2D Burgers equation

$$u_{tx} - (u^2)_{xx} - u_{xxx} + su_{yy} = 0 \tag{3}$$

to the bilinear form:

$$f_{tx}f - f_tf_x - (f_{xxx}f - f_{xx}f_x) + s(f_{yy}f - f_yf_y) = 0. \tag{4}$$

0305-4470/93/216027+05$07.50

Substitution of the following form:

$$f=1+\exp(z) \tag{5a}$$

$$z=kx+ly-wt+z_0 \tag{5b}$$

into (4) gives the travelling wave solution of (3) as follows:

$$u=k/2+(k/2)\tanh((kx+ly-wt+z_0)/2) \tag{6}$$

where k, l (the wavenumbers in the x and y direction, respectively) and z_0 are arbitrary real constants. The parameter w (the frequency) is determined by the following dispersion relation:

$$-wk-k^3+sl^2=0. \tag{7}$$

In order to search for a travelling wave solution of 2D KdV–Burgers equation (1) using a computer algebra system such as *Mathematica*, we construct a transformation

$$u=q[\log(f)]_{xx}+p[\log(f)]_x. \tag{8}$$

The transformation (8) reduces (1) to the following homogeneous equation in $f(x, y, t)$:

$$\begin{aligned}
&[-72nsf_y^2f_x^2-72nf_tf_x^3+(72/25)m^2f_x^4+(432/5)mnf_x^3f_{xx}+216n^2f_x^2f_{xx}^2\\
&\quad-288n^2f_x^3f_{xxx}]+[-(24/5)msf_y^2f_x-(24/5)mf_tf_x^2+24nsf_{yy}f_x^2+72nf_x^2f_{xt}\\
&\quad+96nsf_yf_xf_{xy}+24nsf_y^2f_{xx}+72nf_tf_xf_{xx}-(432/5)mnf_xf_{xx}^2-72n^2f_{xx}^3\\
&\quad-(432/5)mnf_x^2f_{xxx}+216n^2f_x^2f_{xxxx}]f+[(12/5)msf_{yy}f_x+(24/5)mf_xf_{xt}\\
&\quad+(24/5)msf_yf_{xy}-24nsf_{xy}^2-24nsf_xf_{xyy}+(12/5)mf_tf_{xx}-12nsf_{yy}f_{xx}\\
&\quad-36nf_{xt}f_{xx}-(36/25)m^2f_{xx}^2-36nf_xf_{xxt}\\
&\quad-24nsf_yf_{xxy}-12nf_tf_{xxx}-(96/25)m^2f_xf_{xxx}\\
&\quad+(288/5)mnf_{xx}f_{xxx}+24n^2f_{xxx}^2+(216/5)mnf_xf_{xxx}-36n^2f_{xx}f_{xxxx}\\
&\quad-72n^2f_xf_{xxxxx}]f^2+[-(12/5)msf_{xyy}-(12/5)mf_{xxt}+12nsf_{xxyy}+12nf_{xxxt}\\
&\quad+(12/5)m^2f_{xxxx}-(72/5)mnf_{xxxxx}+12n^2f_{xxxxxx}]f^3=0
\end{aligned} \tag{9}$$

where constants q, p outside the derivative in (8) are adjusted to suit the constant outside the nonlinear term in (1) such that terms of degree 5 and 6 in the derivatives of f vanish; in our case $q=12n$, $p=-12m/5$.

Substituting the expression (5) into (9), we find that the expression (5) is really the solution of (9), provided that k, l and w satisfy the following equations:

$$-25l^2ns+25knw+k^2m^2+30k^3mn-25k^4n^2=0 \tag{10a}$$

$$-l^2ms+kmw+30kl^2ns-30k^2nw-36k^4mn+30k^5n^2=0 \tag{10b}$$

$$15l^2ms-15kmw-175kl^2ns+175k^2nw-11k^3m^2+210k^4mn-175k^5n^2=0 \tag{10c}$$

$$(5kn-m)(-kw+l^2s-k^3m+k^4n)=0 \tag{10d}$$

Solving (10) using *Mathematica*, yields

$$k=\pm m/(5n) \qquad w=[-6m^4\pm 625l^2n^3s]/(125mn^2) \tag{11}$$

and l is an arbitrary constant.

From (5), (8) and (11), we obtain the exact solutions of the 2D KdV–Burgers equation (1) as follows:

$$u_{\text{KdV-B}}(x, y, t)=(3m^2/25n)[\text{sech}^2(z/2)\mp 2\tanh(z/2)\mp 2] \tag{12}$$

where

$$z=\pm(m/5n)x+ly-(-6m^4\pm 625l^2n^3s)t/(125mn^2)+z_0 \tag{13}$$

and l is an arbitrary constant.

Taking note of the fact that the 2D Burgers equation

$$(u_t+uu_x-au_{xx})_x+su_{yy}=0$$

and the 2D KdV (KP) equation

$$(u_t+uu_x+bu_{xxx})_x+su_{yy}=0$$

have the following travelling wave solutions, respectively [7, 8]:

$$u(x, y, t)=-ak[\tanh(r/2)+1]$$

$$u(x, y, t)=3bk^2[\text{sech}^2(r/2)]$$

where $r=kx+ly-wt+r_0$, k, l, and w satisfy the following dispersion relations, respectively:

$$kw-l^2s+ak^3=0$$

$$kw-l^2s-bk^4=0$$

we can see easily that

$$u_{\text{B}}(x, y, t)=\mp(6m^2/25n)[\tanh(z/2)+1]$$

and

$$u_{\text{KdV}}(x, y, t)=\mp(18m^2/25n)[\text{sech}^2(z/2)]$$

are the solutions of the equations:

$$(u_t+uu_x-(6m/5)u_{xx})_x+su_{yy}=0 \tag{14}$$

$$(u_t+uu_x\mp 6nu_{xxx})_x+su_{yy}=0 \tag{15}$$

respectively, where z is expressed by (13). Hence we have

$$u_{\text{KdV-B}}(x, y, t)=u_{\text{B}}(x, y, t)\mp(1/6)u_{\text{KdV}}(x, y, t). \tag{16}$$

The fact given above is an important phenomenon, and the results are summed up as follows:

(i) The 2D KdV–Burgers equation (1) has travelling wave solutions (12), which consist of the shock wave solutions of the 2D Burgers equation (14) and the solitary wave solutions of the 2D KdV equation (15).

(ii) Computer algebra systems are extremely useful tools, especially when used to solve computationally tedious problems. In our application, we used *Mathematica* to find a class of exact solution of a nonlinear partial differential equation.

Appendix

Below is a *Mathematica* session used to verify our results. After making an identical transformation, solution (12) and (13) can be expressed in the following form

$$u_1(x, y, t) = -12m^2/(25n)/(1+\exp(-z))^2 \tag{A1a}$$

$$z = (m/5n)x + ly - (-6m^4 + 625ln^3s)t/(125mn^2) + z_0 \tag{A1b}$$

$$u_2(x, y, t) = 12m^2/(25n)[1 - 1/(1+\exp(z))^2] \tag{A2a}$$

$$z = (-m/5n)x + ly - (-6m^4 - 625ln^3s)t/(125mn^2) + z_0. \tag{A2b}$$

We first define two functions $u1[x, y, t]$ and $u2[x, y, t]$ in In[1] and In[2], respectively, corresponding to (A1) and (A2), where m, n, k, l, w and s are all constants. We then ask *Mathematica* to evaluate and differentiate the expression $u1[x, y, t]$ and $u2[x, y, t]$ in the command line In[3] and In[4]. It takes some 1 CPU minute on a Donghai 386 to simplify the results. Taking $k = \pm m/(5n)$, and $w = (-6m^4 \pm 625l^2n^3s)/(125mn^2)$, the final results are as expected, namely 0.

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  In[1]:=u1[x, y, t]:=-12m^2/(25n)/(1+exp[-(k*x+l*y-w*t+z0)])^2
  In[2]:=u2[x, y, t]:=12m^2/(25n)*(1-1/(1+exp[k*x+l*y-w*t+z0])^2)
  In[3]:=zb1=D[D[u1[x, y, t], t]
            +u1[x, y, t]*D[u1[x, y, t], x]-m*D[u1[x, y, t], {x, 2}]
            +n*D[u1[x, y, t], {x, 3}], x]+s*D[u1[x, y, t], {y, 2}]
Time=0.33s
  In[4]:=zb2=D[D[u2[x, y, t], t]
            +u2[x, y, t]*D[u2[x, y, t], x]-m*D[u2[x, y, t], {x, 2}]
            +n*D[u2[x, y, t], {x, 3}], x]+s*D[u2[x, y, t], {y, 2}]
Time=0.32s
  In[5]:=res1=Simplify[zb1]
Time=59.81s
  In[6]:=res2=Simplify[zb2]
Time=56.79s
  In[7]:=Simplify[res1/.{k->m/(5n), w->(-6m^4
            +625l^2*n^3*s)/(125m*n^2)}]
  Out[7]:=0
Time=7.03s
  In[8]:=Simplify[res2/.{k->-m/(5n), w->(-6m^4
            -625l^2*n^3*s)/(125m*n^2)}]
  Out[8]:=0
Time=7.14s
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